ABOUT THE USE OF NUMERICAL, ANALYTICAL BOUNDARY ELEMENT METHOD TO CALCULATE ANISOTROPIC PLATE

Mykola Surianinov*, Vitalii Chaban*

Shown an approach to the calculation of anisotropic plates numerically-analytical boundary elements method. The two-dimensional problem is reduced to one-dimensional by variation method Kantorovich-Vlasov. To select a function of the transverse distribution of deflections are encouraged to use one of two methods – dynamic or static. Application of numerical and analytical boundary element method allows a single approach to obtain the solution of basic differential equation of bending of anisotropic plate with any boundary conditions and without any restrictions on the nature of the application of the external load.

Keywords: *anisotropic plate, boundary elements method, method Kantorovich-Vlasov, characteristic equation, deflection*

Calculation of structures of anisotropic materials and, in particular, the calculation of anisotropic plates, coupled with mathematical difficulties, so to obtain an analytical solution of the differential equation of bending of anisotropic plate is not always possible. An important role in this is played by the fixing plate edges and local load. Widely used numerical methods of analysis, but here, as we know, there is no universal approach. Each numerical method due to the need to solve a specific range of tasks, and having certain advantages, is not without flaws, often of a fundamental nature, that determine the boundaries of its application.

On this ground it is effective to use numerical-analytical boundary elements method (NA BEM), which was developed relatively recently, but has already proven itself for a solution wide range applications [1, 2]. This method allowed to receive a fundamental system of solutions of the differential equation of bending isotropic plates without any restrictions on the nature of the load and secure conditions [1, 2]. Here we consider the idea of spreading the method for calculation of bending anisotropic plates.

The differential equation of bending of anisotropic plate (Fig. 1) has the form [3].

$$\begin{aligned} D_{11}\,\frac{\partial^4 W(x,y)}{\partial x^4} + 4\,D_{16}\,\frac{\partial^4 W(x,y)}{\partial x^3\,\partial y} + 2\,(D_{12}+2\,D_{66})\,\frac{\partial^4 W(x,y)}{\partial x^2\,\partial y^2} &+ \\ + 4\,D_{26}\,\frac{\partial^4 W(x,y)}{\partial x\,\partial y^3} + D_{22}\,\frac{\partial^4 W(x,y)}{\partial y^4} &= q(x,y)\ , \end{aligned} \tag{1}$$

where D_{11}, D_{22} – flexural stiffness relative to the axes y and x; D_{66} – torsional stiffness; D_{16}, D_{26} – external stiffness.

* Prof. Dr. M. Surianinov, Prof. Dr. V. Chaban, Kyiv National University of Technologies and Design, Nemirovich-Danchenko Street, 2, Kyiv 01011, Ukraine

The main governing equation (1) has the fourth order and a differential equation in partial derivatives. The function that is a solution of this equation depends on two variables that mean there is a two-dimensional problem. As known [1, 2], a major element in the sampling system in the numerical-analytical boundary elements method is a one-dimensional module (for rod systems) or generalized dimensional module (for plates and shells), so the equation (1) should be transformed. For this we use the variation Kantorovich-Vlasov method [1].

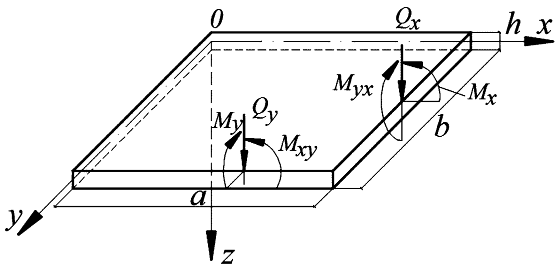


Fig.1: Anisotropic plate

Lets expand the deflection $W(x, y)$ in a functional series:

$$W(x, y) = W_1(y)\, X_1(x) + W_2(y)\, X_2(x) + \cdots + W_n(y)\, X_n(x) \ . \tag{2}$$

The dimensionless system $X_i(x)$ function must be selected such that it most accurately describes the shape of the curved surface of the plate in the direction of the axis ox. Clearly, this requirement is satisfied by beam deflection curves having the same support conditions as the plate in the axis ox direction. To select a function of the transverse distribution of deflections $X(x)$ there are two ways – the static and dynamic [1]. When using a static deflection of the beam is determined by the method of static load (Fig. 2). This load should be such that consistently alternated symmetric and skew shape of the curve deflection. Function $X_i(x)$ presented in the form of power polynomials, which are easy to differentiate, integrate and calculate without the use of complex programs. When using dynamic beam deflection method represents a form of its own oscillations (Fig. 3). If a static method is necessary to build a function $X_i(x)$ depending on the load and the reactions of the beam, the dynamic method is sufficient to change only the values of the natural frequencies, which is very convenient. Functions $X_1(x)$ ('1' index these functions hereinafter omitted) for various support conditions are shown in Table 1.

We hold in (2) one member of the series, which, as shown in our previous works [1, 2], provides acceptable for engineering calculations accuracy of the final result:

$$W(x, y) = W(y)\, X(x) \ . \tag{3}$$

In calculation practice rarely use two or more members of the series (2), limited to the first approximation.

This is due to the high accuracy of the results, due to slight differences between the approximate scheme and the real object. Formally, this is expressed in the proper function $X(x)$ selection. The more accurately it describes a parameter in the direction of the axis ox, the less error of the result.

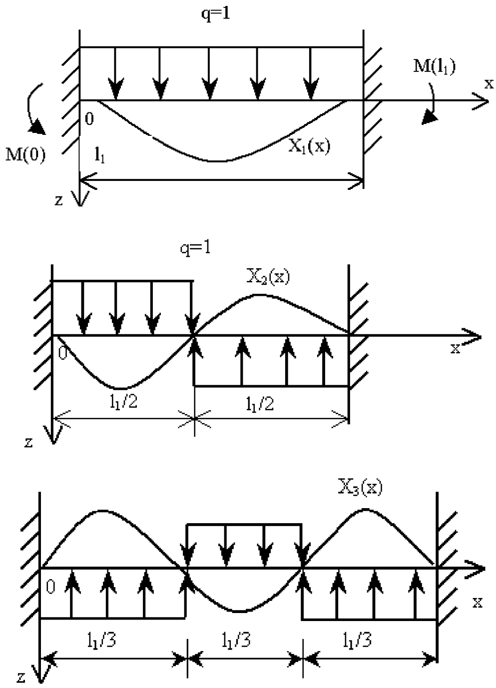


Fig.2: Deflection functions (static)

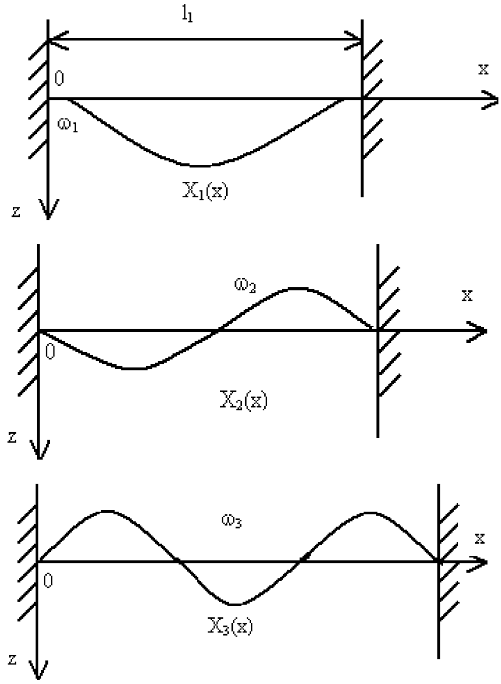
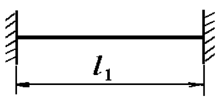
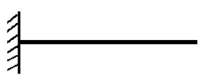
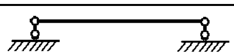
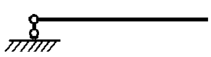


Fig.3: Deflection functions (dynamic)

Scheme of beams	Form of natural oscillations
	$X(x) = \sin\frac{\omega\,x}{l_1} - \sinh\frac{\omega\,x}{l_1} - \alpha_*\left[\cos\frac{\omega\,x}{l_1} - \cosh\frac{\omega\,x}{l_1}\right]\,,$ $\alpha_* = \frac{\sin\omega - \sinh\omega}{\cos\omega - \cosh\omega}$
	$X(x) = \sin\frac{\omega\,x}{l_1} - \sinh\frac{\omega\,x}{l_1} - \alpha_*\left[\cos\frac{\omega\,x}{l_1} - \cosh\frac{\omega\,x}{l_1}\right]\,,$ $\alpha_* = \frac{\sin\omega + \sinh\omega}{\cos\omega + \cosh\omega}$
	$X(x) = \sin\frac{\omega\,x}{l_1} - \sinh\frac{\omega\,x}{l_1} - \alpha_*\left[\cos\frac{\omega\,x}{l_1} - \cosh\frac{\omega\,x}{l_1}\right]\,,$ $\alpha_* = \frac{\sin\omega + \sinh\omega}{\cos\omega + \cosh\omega}$
	$X(x) = \sin\frac{\omega\,x}{l_1}$
	$X(x) = \sin\frac{\omega\,x}{l_1} + \alpha_*\,\sinh\frac{\omega\,x}{l_1}\,,$ $\alpha_* = \frac{\sin\omega}{\sinh}$
	$X(x) = \sin\frac{\omega\,x}{l_1} + \sinh\frac{\omega\,x}{l_1} - \alpha_*\left[\cos\frac{\omega\,x}{l_1} + \cosh\frac{\omega\,x}{l_1}\right]\,,$ $\alpha_* = \frac{\sin\omega - \sinh\omega}{\cos\omega - \cosh\omega}$

Tab.1: Functions $X_1(x)$ for different options of bearing

The convergence of the series (2) is due to the fact that the $W(x,y)$ deflection and the $q(x,y)$ right side (it is also expanded in the orthogonal $X_i(x)$ function system) throughout the region occupied by the plate satisfy the Dirichlet conditions that mean, have a final number of discontinuities of the 1st kind and a final number of maxima and minima.

Lets substitute (3) in (1):

$$D_{11}\,X^{\mathrm{IV}}\,W+4\,D_{16}\,X'''\,W'+2\,(D_{12}+2\,D_{66})\,X''\,W''+4\,D_{26}\,X'\,W'''+D_{22}\,X\,W^{\mathrm{IV}}=q\;. \quad (4)$$

Multiply both sides of (4) on X and integrate in $[0,l_1]$ within, which – the size of the plate in the direction of the axis x:

$$\begin{aligned} D_{11}\,W\int_0^{l_1} X^{\mathrm{IV}}\,X\,\mathrm{d}x+4\,D_{16}\,W'\int_0^{l_1} X'''\,X\,\mathrm{d}x+2\,(D_{12}+2\,D_{66})\,W''\int_0^{l_1} X''\,X\,\mathrm{d}x+\\ +4\,D_{26}\,W'''\int_0^{l_1} X'\,X\,\mathrm{d}x+D_{22}\,W^{\mathrm{IV}}\int_0^{l_1} X^2\,\mathrm{d}x=\int_0^{l_1} q(x,y)\,X\,\mathrm{d}x\;. \end{aligned} \quad (5)$$

Lets denote

$$\begin{aligned} &A=D_{22}\int_0^{l_1} X^2\,\mathrm{d}x\;, \quad B=2\,(D_{12}+2\,D_{66})\int_0^{l_1} X''\,X\,\mathrm{d}x\;, \quad C=D_{11}\int_0^{l_1} X^{\mathrm{IV}}\,X\,\mathrm{d}x\;,\\ &K=4\,D_{26}\int_0^{l_1} X'\,X\,\mathrm{d}x\;, \quad L=4\,D_{16}\int_0^{l_1} X'''\,X\,\mathrm{d}x\;, \quad q(y)=\int_0^{l_1} q(x,y)\,X\,\mathrm{d}x\;. \end{aligned} \quad (6)$$

Then equation (5) gets form:

$$A\,W^{\mathrm{IV}}+K\,W'''+B\,W''+L\,W'+C\,W=q(y)\;,$$

or

$$W^{\mathrm{IV}}+a\,W'''+b\,W''+c\,W'+d\,W=\frac{1}{A}\,q(y)\;, \quad (7)$$

where $a=K/A$, $b=B/A$, $c=L/A$, $d=C/A$.

Coefficients A, B, C, K, L can be calculated (for known rigidities) in any mathematical package, for example, in MATLAB.

Characteristic equation that appropriate equation:

$$t^4+a\,t^3+b\,t^2+c\,t+d=0\;. \quad (8)$$

Equation (8) is algebraic equation forth degree, for which exist analytical solution in radicals for any values of coefficients. There are known solution Descartes-Euler and Ferrari that were detail described in mathematical literature.

Now consider the internal force factors that arising on flexing of anisotropic plate (Fig. 1).

Expressions that determining these factors are well known [3]:

$$M_{\mathrm{x}}=-\left(D_{11}\,\frac{\partial^2 W}{\partial x^2}+D_{12}\,\frac{\partial^2 W}{\partial y^2}+2\,D_{16}\,\frac{\partial^2 W}{\partial x\,\partial y}\right)\;, \quad (9)$$

$$M_{\mathrm{y}} = -\left(D_{12}\,\frac{\partial^2 W}{\partial x^2} + D_{22}\,\frac{\partial^2 W}{\partial y^2} + 2\,D_{26}\,\frac{\partial^2 W}{\partial x\,\partial y}\right)\;, \tag{10}$$

$$M_{\mathrm{xy}} = -\left(D_{16}\,\frac{\partial^2 W}{\partial x^2} + D_{26}\,\frac{\partial^2 W}{\partial y^2} + 2\,D_{66}\,\frac{\partial^2 W}{\partial x\,\partial y}\right)\;, \tag{11}$$

$$Q_{\mathrm{x}} = -\left[D_{11}\,\frac{\partial^3 W}{\partial x^3} + 3\,D_{16}\,\frac{\partial^3 W}{\partial x^2\,\partial y} + (D_{12} + 2\,D_{66})\,\frac{\partial^3 W}{\partial x\,\partial y^2} + D_{26}\,\frac{\partial^3 W}{\partial y^3}\right]\;, \tag{12}$$

$$Q_{\mathrm{y}} = -\left[D_{16}\,\frac{\partial^3 W}{\partial x^3} + (D_{12} + 2\,D_{66})\,\frac{\partial^3 W}{\partial x^2\,\partial y} + 3\,D_{26}\,\frac{\partial^3 W}{\partial x\,\partial y^2} + D_{22}\,\frac{\partial^3 W}{\partial y^3}\right]\;. \tag{13}$$

Normal and shear tensions (Fig. 4) are connected to the internal power dependencies:

$$\begin{gathered}\sigma_{\mathrm{x}} = \frac{12\,M_{\mathrm{x}}}{h^3}\,z\;, \qquad \sigma_{\mathrm{y}} = \frac{12\,M_{\mathrm{y}}}{h^3}\,z\;, \qquad \tau_{\mathrm{xy}} = \frac{12\,M_{\mathrm{xy}}}{h^3}\,z\;, \\ \tau_{xy} = \tau_{zx} = \frac{6\,Q_x}{h^3}\left(\frac{h^2}{4} - z^2\right)\;, \qquad \tau_{yz} = \tau_{zy} = \frac{6\,Q_y}{h^3}\left(\frac{h^2}{4} - z^2\right)\;.\end{gathered} \tag{14}$$

Expressions (9)–(13) are not suitable for use in the numerical-analytical method of boundary elements, because they are functions with two variables, and method is considered one-dimensional modules. We apply to (9)–(13) variation method of Kantorovich-Vlasov.

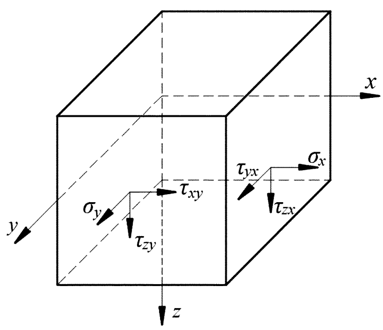


Fig.4: Normal and shearing stresses in plate

Substituting (2) in each of the expressions (9)–(13):

$$M_x = -(D_{11}\,W\,X'' + D_{12}\,W''\,X + 2\,D_{16}\,W'\,X')\;, \tag{15}$$

$$M_y = -(D_{12}\,W\,X'' + D_{22}\,W''\,X + 2\,D_{26}\,W'\,X')\;, \tag{16}$$

$$M_{xy} = -(D_{16}\,W\,X'' + D_{26}\,W''\,X + 2\,D_{66}\,W'\,X')\;, \tag{17}$$

$$Q_x = -[D_{11}\,W\,X''' + 3\,D_{16}\,W'\,X'' + (D_{12} + 2\,D_{66})\,W''\,X' + D_{26}\,W'''\,X]\;, \tag{18}$$

$$Q_y = -[D_{16}\,W\,X''' + (D_{12} + 2\,D_{66})\,W'\,X'' + 3\,D_{26}\,W''\,X' + D_{22}\,W'''\,X]\;. \tag{19}$$

In formulas (15)–(19) $W = W(y)$, $X = X(x)$.

Supplement (15)–(19) kinematic parameters (rotation angles)

$$\begin{gathered}\theta_{\mathrm{x}} = W\,X'\;, \\ \theta_{\mathrm{y}} = W'\,X\;.\end{gathered} \tag{20}$$

The plate is regarded as a generalized one-dimensional module, so its state vector is the same as in the bending beam :

$$\vec{F} = \begin{vmatrix} W(y) \\ \theta_{\mathrm{y}}(y) \\ M_{\mathrm{y}}(y) \\ Q_{\mathrm{y}}(y) \end{vmatrix} , \tag{21}$$

where W, θ_{y}, M_{y}, Q_{y} – deflection, angle of rotation, bending moment and shear force respectively.

Solution of equation (7) depends on the root of the corresponding characteristic equation (8), which, as shown by S. G. Lehnitsky [3] can not have real roots

$$t_{1-4} = \pm\alpha \pm \mathrm{i}\,\beta \ .$$

Therefore, the deflection can be written as

$$W(y) = C_1\,\Phi_1 + C_2\,\Phi_2 + C_3\,\Phi_3 + C_4\,\Phi_4 \ , \tag{22}$$

where

$$\begin{aligned} \Phi_1 &= \cosh\alpha y\,\sin\beta y \ , \qquad & \Phi_2 &= \cosh\alpha y\,\cos\beta y \ , \\ \Phi_3 &= \sinh\alpha y\,\cos\beta y \ , \qquad & \Phi_4 &= \sinh\alpha y\,\sin\beta y \ . \end{aligned} \tag{23}$$

After determining the constants by solving the Cauchy problem we can determine the fundamental functions of the problem, to construct the Green's function, etc. in accordance with the algorithm of numerical and analytical boundary elements method, which is detail described in [1, 6].

References

[1] Dashchenko A.F., Kolomiets L.V., Orobie V.F., Suryaninov N.G., Numerical-analytical boundary elements method, Odessa: VMV, 2010, In 2 vols.: Vol. 1 – 416 p., Vol. 2, 512 p.
[2] Orobie V.F. Suryaninov N.G.: Workshop on the solution of boundary value problems of mechanics: Textbook for students of technical specialties, Odessa: Astroprint, 2011, 408 p.
[3] Lehnitsky S.G.: Anisotropic Plates, M.L.: OGIZ GITTL, 1947, 355 p.
[4] Korn G., Korn T.: Handbook of mathematics for scientists and engineers, M.: Nauka, 2003, 832 p.
[5] Suryaninov N.G.: Appendix numerical-analytical boundary element method for calculation of orthotropic plates, N.G. Suryaninov, I.V. Pavlenko, Pratsi Odeska polytehnichnogo universitetu Naukova that NAUKOVO-virobnichy zbirnik, Odessa, 2014, Vol. 1 (43), 18–27
[6] Orobie V.F., Suryaninov N.G.: Summary of numerical and analytical option BEM, Proceedings of the St. Perburgskogo politehnich. Univ, Civil Engineering Journal 4 (22), St. Petersburg, 2011, 33–39

Received in editor's office: July 7, 2015
Approved for publishing: September 29, 2015