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Обмежені розв'язки різницевого рівняння другого порядку зі стрибками операторних коефіцієнтів

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Bounded solutions of a second order difference equation with jumps of operator coefficient

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В роботі вивчаєтьcя питання існування единого обмеженого розв'язку різницевого рівняння другого порядку зі змінним операторним коефіцієнтом у банаховому просторі. Для випадку скінченного числа стрибків операторного коефіцієнта отримано необхідні та достатні умови.
Ключові слова: Різницеве рівняння, обмежений розв'язок, банахів простір .

We study the problem of existence of a unique bounded solution of a difference equation of the second order with a variable operator coefficient in a Banach space. In the case of a finite number of jumps of an operator coefficient necessary and sufficient conditions are obtained.
Key Words: Difference equation, bounded solution, Banach space.

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1 Introduction

Let $(X, ||\cdot||)$ be a complex Banach space, $L(X)$ be the space of linear continuous operators in X, $I \in L(X)$ be the identity operator. Let denote as $\sigma(A)$ the spectrum of an operator $A \in L(X)$. Let denote as $S = \{z \in \mathbf{C} \mid |z| = 1\}$ the unit circle in the complex plane.

Let us consider the difference equation

$$x_{n+1} = A_n x_n + B_n x_{n-1} + y_n,\ n \in \mathbf{Z}, \qquad (1)$$

where $\{A_n \mid n \in \mathbf{Z}\} \subset L(X)$, $\{B_n \mid n \in \mathbf{Z}\} \subset L(X)$, $\{y_n \mid n \in \mathbf{Z}\} \subset X$ are known sequences, $\{x_n \mid n \in \mathbf{Z}\} \subset X$ is a desired sequence. In the paper we investigate the question of existence and uniqueness of a bounded solution to the equation (1).

It is known [1, chapter 7.6] that the equation (1) of the first order has a unique bounded solution $\{x_n \mid n \in \mathbf{Z}\}$ for any bounded sequence $\{y_n \mid n \in \mathbf{Z}\}$ iff an operators sequence fulfills a condition of discrete dichotomy (by analogy with an exponential dichotomy which is well known in the theory of differential equations). However, checking of discrete dichotomy conditions is very hard, so we need simpler conditions of existence and uniqueness of a bounded solution for special operators sequences.

I.V.Gonchar and M.F.Gorodnii investigated the equation (1) in the papers [3, 4] for the case of first order and one jump of an operator coefficient.

In the article [5] the equation (1) was investigated in the case of the first order and several jumps.

To formulate main obtained results we need the following spectral decomposition. Assume $A \in L(X)$ and the condition

$$\sigma(A) \cap S = \varnothing$$

is true. Then the spectrum of the operator A is decomposed into two parts, one of them is inside of the unit circle S, the other is outside. Using the theorem about decomposition [2, p. 445], we can derive:

1) an existence of projectors $P_-(A), P_+(A) \in L(X)$ such that

$$P_-(A) + P_+(A) = I;$$

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2) decomposition of the space X to the direct sum

$$X = X_-(A)\dot{+}X_+(A), \tag{2}$$

where

$$X_-(A) = P_-(A)X,\ X_+(A) = P_+(A)X$$

are subspaces in which corresponding operators

$$A_- = P_-(A)A,\ A_+ = P_+(A)A$$

have spectra

$$\begin{array}{c} \sigma(A_-) = \sigma(A) \cap \{z \in \mathbf{C} \mid |z| < 1\}, \\ \sigma(A_+) = \sigma(A) \cap \{z \in \mathbf{C} \mid |z| > 1\} \end{array} \tag{3}$$

accordingly.

In the paper [4] the following result was proved.

Theorem 1.1. *Let X be a complex Banach space and G, U be some operators from $L(X)$, which satisfy the following conditions:*

1) $\sigma(G) \cap S = \varnothing,\ \sigma(U) \cap S = \varnothing$;

2) $X = X_-(G)\dot{+}X_+(U)$.

Then the difference equation

$$\begin{cases} x_{n+1} = Gx_n + y_n, & n \geqslant 1, \\ x_{n+1} = Ux_n + y_n, & n \leqslant 0, \end{cases}$$

has a unique bounded in X solution $\{x_n \mid n \in \mathbf{Z}\}$ for any bounded in X sequence $\{y_n \mid n \in \mathbf{Z}\}$.

In [5] this result was generalized to the case of a first order equation with several operator jumps

$$\begin{cases} x_{n+1} = A_0x_n + y_n, & n \leqslant 0, \\ x_{n+1} = A_nx_n + y_n, & 1 \leqslant n \leqslant N-1, \\ x_{n+1} = A_Nx_n + y_n, & n \geqslant N. \end{cases} \tag{4}$$

Assume the conditions $\sigma(A_0) \cap S = \varnothing$, $\sigma(A_N) \cap S = \varnothing$ are true. Then each of the operators A_0, A_N produces spectral decomposition of the form (2). Let us denote

$$P_{0-} := P_-(A_0),\ P_{0+} := P_+(A_0),$$

$$P_{N-} := P_-(A_N),\ P_{N+} := P_+(A_N),$$

$$X_{0-} := X_-(A_0),\ X_{0+} := X_+(A_0),$$

$$X_{N-} := X_-(A_N),\ X_{N+} := X_+(A_N).$$

Theorem 1.2. *Let $\sigma(A_0) \cap S = \varnothing, \sigma(A_N) \cap S = \varnothing$ and $A_{N-1}A_{N-2} \cdot \ldots \cdot A_1$ is injection. Then the equation (5) has a unique bounded solution $\{x_n \mid n \in \mathbf{Z}\} \subset X$ for any bounded sequence $\{y_n \mid n \in \mathbf{Z}\} \subset X$ iff*

$$X = W\dot{+}X_{N-},$$

where

$$W = \{A_{N-1}A_{N-2} \cdot \ldots \cdot A_1 x \mid x \in X_{0+}\}.$$

In this article we consider a generalization of this result for a second order equation with an operator coefficient which changes a finite number of times:

$$\begin{cases} x_{n+1} = A_0x_n + B_0x_{n-1} + y_n, & n \leqslant 0, \\ x_{n+1} = A_nx_n + B_nx_{n-1} + y_n, & 1 \leqslant n \leqslant N-1, \\ x_{n+1} = A_Nx_n + B_Nx_{n-1} + y_n, & n \geqslant N. \end{cases} \tag{5}$$

In the paper the result of the theorem 1.2 is generalized to a second order equation (5).

2 Main results

First we rewrite our equation in the space X^2, where norm is defined as

$$||(x_1, x_2)|| = \sqrt{||x_1||^2 + ||x_2||^2}.$$

Lemma 1. *The equation (5) has a unique bounded solution $\{x_n \mid n \in \mathbf{Z}\} \subset X$ for any bounded sequence $\{y_n \mid n \in \mathbf{Z}\} \subset X$ iff an equation*

$$u_{n+1} = C_nu_n + v_n,\ n \in \mathbf{Z}, \tag{6}$$

where

$$C_n := \begin{pmatrix} A_n & B_n \\ I & O \end{pmatrix},\ n \in \mathbf{Z},$$

has a unique bounded solution $\{u_n \mid n \in \mathbf{Z}\} \subset X^2$ for any bounded sequence $\{v_n \mid n \in \mathbf{Z}\} \subset X^2$.

Proof. Necessity. For any bounded sequence $\{v_n = (v_n^1, v_n^2) \mid n \in \mathbf{Z}\} \subset X^2$ we have system

$$\begin{cases} u_{n+1}^1 = A_nu_n^1 + B_nu_n^2 + v_n^1,\ n \in \mathbf{Z}, \\ u_{n+1}^2 = u_n^1 + v_n^2,\ n \in \mathbf{Z}. \end{cases}$$

Since equation

$$u_{n+1}^1 = A_nu_n^1 + B_nu_{n-1}^1 + B_nv_{n-1}^2 + v_n^1,\ n \in \mathbf{Z}$$

has a unique bounded solution, so the system has too.

Sufficiency. Let for any bounded sequence $\{y_n \mid n \in \mathbf{Z}\} \subset X$ put $v_n = (y_n, 0)$, $n \in \mathbf{Z}$. Then there is a unique bounded solution $\{u_n = (u_n^1, u_n^2) \mid n \in \mathbf{Z}\} \subset X^2$ of the equation (6) which is equivalent to a system

$$\begin{cases} u_{n+1}^1 = A_n u_n^1 + B_n u_n^2 + y_n, \ n \in \mathbf{Z}, \\ u_{n+1}^2 = u_n^1, \ n \in \mathbf{Z}. \end{cases}$$

So, $\{x_n = u_n^1 \mid n \in \mathbf{Z}\} \subset X$ is the unique bounded solution of (5). □

Let

$$C_n := \begin{pmatrix} A_n & B_n \\ I & O \end{pmatrix}, \ n \in \mathbf{Z}.$$

If for some $n \in \mathbf{Z}$ we have $\sigma(C_n) \cap S = \varnothing$, let denote as V_{n+} the subspace of X^2, corresponding to the part of the spectra which is situated outside of S in the spectral decomposition for C_n and V_{n-} the subspace X^2, corresponding to the part of the spectra which is situated inside of S in the spectral decomposition.

Theorem 2.1. *Let*

$$\forall z \in S \ \exists (zA_0 - z^2 I + B_0)^{-1} \in L(X),$$

$$\exists (zA_N - z^2 I + B_N)^{-1} \in L(X)$$

and

$$Ker(C_{N-1} C_{N-2} \cdot ... \cdot C_1) \cap V_{0+} = \{\vec{0}\}.$$

Then the equation (5) has a unique bounded solution $\{x_n \mid n \in \mathbf{Z}\} \subset X$ for any bounded sequence $\{y_n \mid n \in \mathbf{Z}\} \subset X$ iff

$$X^2 = W \dot{+} V_{N-},$$

where

$$W = \{C_{N-1} C_{N-2} \cdot ... \cdot C_1 x \mid x \in V_{0+}\}.$$

Proof. Using multiplication directly, it is not difficult to be ensured that the resolvents of the operators C_0, C_N in some neighbourhood U of the unit circle S look like

$$R_z(C_n) = \begin{pmatrix} zF_n(z) & zF_n(z)(zI - A_n) + I \\ F_n(z) & zF_n(z)(zI - A_n) \end{pmatrix},$$

$$z \in U, \ n \in \{0, N\}, \qquad (7)$$

where the operators

$$F_n(z) = (zA_n - z^2 I + B_n)^{-1}, \ z \in U, \ n \in \{0, N\}$$

exist because they exist on the unit circle and a resolvent set is open.

Therefore, the spectra of the operators C_0, C_N do not intersect the unit circle, particularly the subspaces V_{N-}, V_{0+} are defined.

Corresponding to the lemma, let consider the equation (6). We can apply the theorem 1.2. Besides injection, all the demands of the theorem are satisfied. However, according to the proof of this theorem in the article [5], we can note that injection was used only for the proof of the lemma 3 and exceptionally for the restriction of the operator $C_{N-1} C_{N-2} \cdot ... \cdot C_1$ on the set V_{0+}. But such a simplified weakened condition is following from the condition of the theorem. □

3 Partial cases

1. Let apply the theorem 2.1 in the case when $B_n = O, \ n \in \mathbf{Z}$.

The condition of the existence of the inverse operators is equivalent to $\sigma(A_0) \cap S = \varnothing, \sigma(A_N) \cap S = \varnothing$. In this case

$$V_{n+} = \{(u, v) \mid u \in X_{n+}, \ v \in X\},$$

$$V_{n-} = \{(u, \vec{0}) \mid u \in X_{n-}\}, \ n = 0, N.$$

Really,

$$C_n^k = \begin{pmatrix} A_n & O \\ I & O \end{pmatrix}^k = \begin{pmatrix} A_n^k & O \\ A_n^{k-1} & O \end{pmatrix}, \ k \in \mathbf{N},$$

therefore,

$$||C_n^k(u, v)|| = ||(A_n^k u, A_n^{k-1} u)|| =$$

$$= \sqrt{||A_{n+}^k u||^2 + ||A_{n+}^{k-1} u||^2} \to +\infty, \ u \in X_{n+} \backslash \{\vec{0}\},$$

$$||C_n^k(u, \vec{0})|| = ||(A_n^k u, A_n^{n-k} u)|| =$$

$$= \sqrt{||A_{n-}^k u||^2 + ||A_{n-}^{n-k} u||^2} \to 0, \ u \in X_{n-}.$$

The condition $Ker(C_{N-1} C_{N-2} \cdot ... \cdot C_1) \cap V_{0+} = \{\vec{0}\}$ means that the kernel of the operator

$$C_{N-1} C_{N-2} \cdot ... \cdot C_1 = \begin{pmatrix} A_{N-1} A_{N-2} \cdot ... \cdot A_1 & O \\ A_{N-2} \cdot ... \cdot A_1 & O \end{pmatrix}$$

does not include the elements from the set $\{(u, v) \mid u \in X_{0+}, \ v \in X\}$, in other words

$$Ker(A_{N-1} A_{N-2} \cdot ... \cdot A_1) \cap X_{0+} = \{\vec{0}\}.$$

Eventually, the condition of a decomposition in a direct sum will look

$$\forall (x_1, x_2) \in X^2 \quad \exists!(w_1, w_2) \in W \quad \exists! v \in X_{N-} \; :$$

$$(x_1, x_2) = (w_1 + v, w_2).$$

Here the operator $C_{N-1}C_{N-2} \cdot \ldots \cdot C_1$ is the injection on V_{0+}, that is why the condition can be rewrited as

$$\forall (x_1, x_2) \in X^2 \;\; \exists! u_1 \in X_{0+} \;\; \exists! u_2 \in X \exists! v \in X_{N-} \; :$$

$$(x_1, x_2) = (A_{N-1}A_{N-2} \cdot \ldots \cdot A_1 u_1 + v, u_2).$$

Summary, the next statement is true.

Corollary 1. *Let* $B_0 = \ldots = B_N = O$, $\sigma(A_0) \cap S = \varnothing, \sigma(A_N) \cap S = \varnothing$ *and*

$$Ker(A_{N-1}A_{N-2} \cdot \ldots \cdot A_1) \cap X_{0+} = \{\vec{0}\}.$$

Then the equation (5) has a unique bounded solution $\{x_n \mid n \in \mathbf{Z}\} \subset X$ *for any bounded sequence* $\{y_n \mid n \in \mathbf{Z}\} \subset X$ *iff*

$$\forall x \in X \quad \exists! u \in X_{0+} \quad \exists! v \in X_{N-} \; :$$

$$x = A_{N-1}A_{N-2} \cdot \ldots \cdot A_1 u + v.$$

This statement is some improvement of the result of the theorem 1.2 in the partial case which is considering.

2. Let apply the theorem 2.1 in the case when $A_n = O, \; n \in \mathbf{Z}$.

The condition of the existence of the inverse operators is equivalent to the condition

$$\sigma(B_0) \cap S = \varnothing, \sigma(B_N) \cap S = \varnothing.$$

Here

$$V_{n+} = X_+(B_n), \; V_{n-} = X_-(B_n), \; n = 0, N.$$

Дійсно,

$$C_n^{2k} = \begin{pmatrix} O & B_n \\ I & O \end{pmatrix}^{2k} = \begin{pmatrix} B_n & O \\ O & B_n \end{pmatrix}^{k} =$$

$$= \begin{pmatrix} B_n^k & O \\ O & B_n^k \end{pmatrix}, \quad k \in \mathbf{N},$$

therefore

$$||C_n^{2k}(u, v)|| = ||(B_n^k u, B_n^k v)|| =$$

$$= \sqrt{||B_{n+}^k u||^2 + ||B_{n+}^k v||^2} \to +\infty,$$

$$u, v \in X_{n+} \backslash \{\vec{0}\},$$

$$||C_n^{2k}(u, v)|| = ||(B_n^k u, B_n^k v)|| =$$

$$= \sqrt{||B_{n-}^k u||^2 + ||B_{n-}^k v||^2} \to 0, \; u, v \in X_{n-}.$$

Note that

$$C_{N-1}C_{N-2} \cdot \ldots \cdot C_1 =$$

$$= \begin{pmatrix} B_{2k}B_{2k-2} \cdot \ldots \cdot B_2 & O \\ O & B_{2k-1}B_{2k-3} \cdot \ldots \cdot B_1 \end{pmatrix}$$

when $N = 2k+1, \; k \in \mathbf{N}$, i

$$C_{N-1}C_{N-2} \cdot \ldots \cdot C_1 =$$

$$= \begin{pmatrix} O & B_{2k-1} \cdot \ldots \cdot B_1 \\ B_{2k-2}B_{2k-4} \cdot \ldots \cdot B_2 & O \end{pmatrix}$$

when $N = 2k, \; k \in \mathbf{N}$.

Therefore, the condition

$$Ker(C_{N-1}C_{N-2} \cdot \ldots \cdot C_1) \cap V_{0+} = \{\vec{0}\}$$

means that the kernels of the operators

$$B_{N-1} \cdot B_{N-3} \cdot \ldots \cdot B_{N+1-2[N/2]},$$

$$B_{N-2} \cdot B_{N-4} \cdot \ldots \cdot B_{N-2[(N-1)/2]}$$

do not include the nontrivial elements of X_{0+}.

Eventually, the condition of decomposition in a direct sum will look

$$\forall (x_1, x_2) \in X^2 \;\; \exists!(w_1, w_2) \in W \;\; \exists!(v_1, v_2) \in X_{N-} \; :$$

$$(x_1, x_2) = (w_1 + v_1, w_2 + v_2),$$

or equivalently

$$\forall x \in X^2 \;\; \exists! w = C_1 u, \; u \in X_{0+}^2 \;\; \exists! v \in X_{N-}^2 \; :$$

$$x = C_{N-1}C_{N-2} \cdot \ldots \cdot C_1 u + v.$$

Here the operator $C_{N-1}C_{N-2} \cdot \ldots \cdot C_1$ is injection on V_{0+}, that is why the condition can be rewrited as a union of two conditions

$$\forall x \in X^2 \;\; \exists! u \in X_{0+} \;\; \exists! v \in X_{N-} \; :$$

$$x = B_{N-1}, B_{N-2} \cdot \ldots \cdot B_{N+1-2[N/2]} u + v,$$

$$\forall x \in X^2 \;\; \exists! u \in X_{0+} \;\; \exists! v \in X_{N-} \; :$$

$$x = B_{N-2} \cdot \ldots \cdot B_{N-2[(N-1)/2]} u + v.$$

So, such a statement appears to be true:

Corollary 2. *Let* $A_0 = ... = A_N = O$, $\sigma(B_0) \cap S = \varnothing, \sigma(B_N) \cap S = \varnothing$ *and*

$$Ker(B_{N-1} \cdot B_{N-3} \cdot ... \cdot B_{N+1-2[N/2]}) \cap X_{0+} = \{\vec{0}\},$$

$$Ker(B_{N-2} \cdot B_{N-4} \cdot ... \cdot B_{N-2[(N-1)/2]}) \cap X_{0+} = \{\vec{0}\}.$$

Then the equation (5) has a unique bounded solution $\{x_n \mid n \in \mathbf{Z}\} \subset X$ *for any bounded sequence* $\{y_n \mid n \in \mathbf{Z}\} \subset X$ *iff*

$$\forall x \in X^2 \quad \exists! u \in X_{0+} \quad \exists! v \in X_{N-} \ :$$

$$x = B_{N-1} \cdot B_{N-3} \cdot ... \cdot B_{N+1-2[N/2]} u + v,$$

and

$$\forall x \in X^2 \quad \exists! u \in X_{0+} \quad \exists! v \in X_{N-} \ :$$

$$x = B_{N-2} \cdot B_{N-4} \cdot ... \cdot B_{N-2[(N-1)/2]} u + v.$$

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